Minimal Model Program

Learning Seminar.

Week 4:

Base point free Theorem.

The Cone Theorem.

Week 4:

bpf Theorem.

The cone Theorem.

The Cone Theorems:

old notation		new notation
(X,Δ) klt	$\Longrightarrow$	(X,Δ) sub-klt
(X,Δ) klt with $\Delta \geq 0$		(X,Δ) klt.

Remark: As of 2021, this still confuses people.

klt.

Theorem (Non-vanishing): X proper, (X,Δ) sub-klt, D nef Cartier Cartier. Assume $aD-(K_X+\Delta)$ is big and nef for some $a \geq 0$. Then for $m \gg 0$

$$H^0(X, mD - \lfloor \Delta \rfloor) \neq 0.$$

$\lfloor \Delta \rfloor = 0$

Theorem (bpf): X proper, (X,Δ) klt, D nef Cartier divisor. Assume $aD-(K_X+\Delta)$ is big and nef for some $a \geq 0$. Then for $m \gg 0$

$$|mD| \text{ is bpf}$$

Theorem (Rationality): X proper, (X,Δ) klt.

$K_X+\Delta$ not nef, $a(K_X+\Delta)$ Cartier. H nef and big Cartier.

Define $r := r(H) = \max\{t\in\mathbb{R} \mid H+t(K_X+\Delta) \text{ is nef}\}$.

Then r is rational and its denom is controlled by $a(\dim X+1)$.

Theorem (Cone Theorem): (X,Δ) projective klt pair.

(1) There are countably many $C_i\subseteq X$ s.t. $0<-(K_X+\Delta).C_i\leq 2\dim X$ &

$$\overline{NE}(X) = \overline{NE}_{(K_X+\Delta)\geq 0} + \sum_i{}' \mathbb{R}_{\geq 0}[C_i]$$

(2) For any $\varepsilon>0$ and H ample.

$$\overline{NE}(X) = \overline{NE}_{(K_X+\Delta+\varepsilon H)\geq 0} + \sum_{\text{finite}}{}' \mathbb{R}_{\geq 0}[C_i].$$

formal cons of Rat + convex geom

(3) $F\subseteq \overline{NE}(X)$ extremal, $(K_X+\Delta)$-neg.

Then there exists a contraction morphism $\mathrm{cont}_F: X\to Z$

$C\subseteq X$ mapped to a point $\iff$ $[C]\in F$.

(4) $\mathrm{cont}_F: X\longrightarrow Z$, as in (3), $\mathcal{L}$ a line bundle on X

s.t. $\mathcal{L}.F=0$. Then there exists $\mathcal{L}_Z$ on Z s.t.

$$\mathcal{L}\simeq \mathrm{cont}_F^*\mathcal{L}_Z.$$

geometric statements.

The Cone Theorems:

Riemann-Roch Theorem.

Non-vanishing

$\Downarrow$ ← Use Kodaira vanishing to lift sections

Base-point free Theorem

$\Downarrow$ ← study linear systems of the form $|pH + qK_X|$ for different (p,q)

Rationality Theorems.

$\Downarrow$ ← formal argument of convex geom

Cone Theorem

Rmk: The whole discussion is with (X,Δ) klt.

But the above theorems hold when (X,Δ) lc

up replacing nef & big $\Longrightarrow$ ample

Proof of bpf Theorem:

By non-vanishing $H^0(X, mD) \neq 0$ for $m \gg 0$.

$B(s)$ the base locus of $|sD|$.

It suffices to prove that for $\boxed{B_s = B(m) \neq \emptyset}$

$$f : Y \longrightarrow X \quad \text{log resolution} \quad K_Y = f^*(K_X+\Delta) + \sum{}' a_j F_j$$

$a_j > -1$.

$$\underbrace{\underbrace{f^*(aD - (K_X+\Delta))}_{\text{big \& nef}} - \sum{}' p_j F_j}_{\text{ample}} \qquad 0 < p_j \ll 1$$

ample over X.

fixed part

$$f^*|mD| = |M| + \sum{}' r_j F_j$$

$B_s = \bigcup \{ f(F_j) \mid r_j > 0 \}$. Note that

$$f^{-1} B_s|mD| = B_s|mf^*D|$$

There exists F_j with $r_j \geq 0$ so that for all $b \gg 0$.

F_j is not contained in $Bs\,|bf^*D|$

$b > 0$ integer, $c \geq 0$ rational $b > cm + a$, we define

$$N(b,c) = bf^*D - K_Y + \sum\nolimits_j' (\underbrace{-cr_j + a_j - p_j}_{> -1}) F_j$$

$$\left.\begin{array}{ll} = (b - cm - a) f^*(D) & \text{(nef)} \\ + & \\ c(mf^*D - \sum\nolimits' r_j F_j) & \text{(bpf)} \\ + & \\ f^*(aD - (K_X + \Delta)) - \sum\nolimits' p_j F_j & \text{(ample)} \end{array}\right\} \text{ample.}$$

$K_Y + A + \text{eff.}$

By Kodaira $H^1(Y, \lceil N(b,c) \rceil + K_Y) = 0$, and

$$\lceil N(b,c) \rceil = bf^*D + \sum\nolimits' \lceil -cr_j + a_j - p_j \rceil F_j - K_Y$$

increase c from 0 to ∞ and wiggle the p_j to achieve

$$\sum_i \lceil -cr_j + a_j - p_j \rceil F_j$$

$$\|$$

$$\underset{\substack{\vee\!\!\vdash \\ 0}}{\lceil A \rceil} - F = F'_j \quad \text{prime}$$

prime

$$K_Y + \lceil N(b,c) \rceil = bf^*D + \lceil A \rceil - F$$

$$0 \longrightarrow \mathcal{O}_Y(bf^*D + \lceil A \rceil - F) \xrightarrow{\times F} \mathcal{O}_Y(bf^*D + \lceil A \rceil) \longrightarrow \mathcal{O}_F(bf^*D + \lceil A \rceil) \longrightarrow 0$$

$$H^0(Y, bf^*D + \lceil A \rceil) \twoheadrightarrow H^0(F, (bf^*D + \lceil A \rceil)|_F)$$

is surjective for $b \geq cm + a$

$\lceil A \rceil$ is f-exceptional.

big & nef

$$N(b,c)|_F = (bf^*D + A - F - K_Y)|_F = (bf^*D + A)|_F - K_F$$

$$H^0(Y, bf^*D + \lceil A \rceil) \twoheadrightarrow \underset{\substack{\neq \\ 0}}{H^0(F, (bf^*D + \lceil A \rceil)|_F)}$$

by Non-vanishing

has a section not vanishing on F.

Since $\underset{\substack{\vee\!/ \\ 0}}{\lceil A \rceil}$ is f-exceptional, we have

$$H^0(Y, bf^*D + \lceil A \rceil) \overset{\subseteq}{\underset{\supseteq}{=}} H^0(Y, bf^*D) = H^0(X, bD).$$

Negativity Lemma: $h: Z \to Y$ birational proper between normal, $-B$ h-nef.

(1) $B \geq 0 \iff h_*B \geq 0$

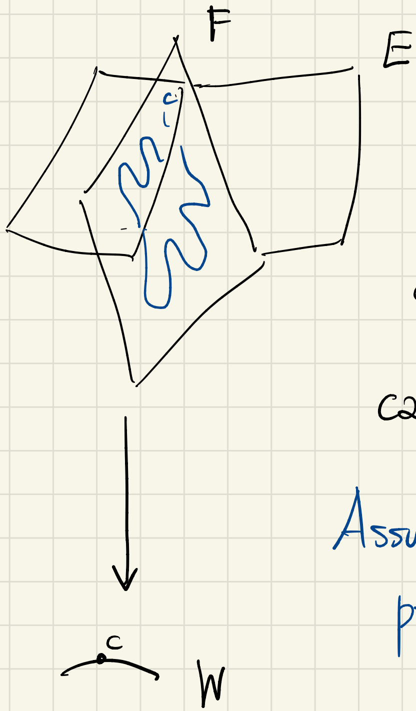
(2) $h^{-1}(y) \subseteq \operatorname{Supp} B$ or $h^{-1}(y) \cap \operatorname{Supp} B = \emptyset$

over the base

$0 \leq E \sim bf^*D + \lceil A \rceil$ $\qquad$ $E - \lceil A \rceil \sim bf^*D \sim_{\mathbb{Q}, X} 0.$

$E - \lceil A \rceil \geq 0$ $\qquad$ $f_*(E - \lceil A \rceil) = f_*E \geq 0$

Now, we have a section E of bf^*D which push-forwards to a section f_*E of bD. We want to argue that the section E is disjoint from F, so f_*E is disjoint from W. This will imply $Bs(bD) \subsetneq Bs(mD)$ contradicting the stabilization of B:



$0 \leq E \sim bf^*D$.

can E int F trans?

Assume C maps to a point in W.

C is general enough on some fiber of f over W.

$C.E > 0$, $\quad C.bf^*D = 0 \quad \Rightarrow\Leftarrow$.

We found a section E of bf^*D disj from F

$\Longrightarrow$ f_*E is a section of bD disj from W. $\square$

Theorem: Let (X,Δ) be proper klt pair. $K_X+\Delta$ big & nef. Then $\bigoplus_{m\geq 0}^{\infty} H^0(\mathcal{O}_X(mK_X+\lfloor m\Delta \rfloor))$ is finitely generated over $\mathbb{C}$.

BCHM06: The finite gen of $R(K_X+\Delta)$

Conjecture (Abundance): (X,Δ) proj klt, if $K_X+\Delta$ is nef, then $K_X+\Delta$ is semiample.

Conjecture (Effectivity): (X,Δ) proj klt, if $K_X+\Delta$ pseff, then $K_X+\Delta$ is eff.

$$K_X+\Delta \sim_{\mathbb{Q}} E \geq 0.$$

The cone theorem.

Theorem: $N_{\mathbb{Z}} \subseteq N_{\mathbb{Q}} \subseteq N_{\mathbb{R}}$. $\overline{NE} \subseteq N_{\mathbb{R}}$ closed strictly convex cone. $K \in N^*_{\mathbb{Q}}$ so that $(K.C) < 0$ for some $C \in \overline{NE}$.

Assume there exists $\alpha(K) \in \mathbb{Z}_{\geq 0}$ such that for all $H \in N^*_{\mathbb{Z}}$ with $H > 0$ on $\overline{NE} - \{0\}$.

$$r := \max \{ t \in \mathbb{R} \mid H + tK \geq 0 \text{ on } \overline{NE} \}.$$

rational of the form $u/\alpha(K)$. Then.

$$\overline{NE} = \overline{NE}_{K \geq 0} + \sum_{\text{countable}} \mathbb{R}_{\geq 0}[\xi_i]$$

$\xi_i \in N_{\mathbb{Z}}$ with $(\xi_i . K) < 0$ and $\mathbb{R}_{\geq 0}[\xi_i]$ do not accumulate in $K_X < 0$.

Proof: Fix H an ample Cartier divisor.

L nef, $F_L = L^\perp \cap NE$, $n \in \mathbb{Z}_{\geq 0}$.

$$r_L(n, H) = \max\left\{ t \in \mathbb{R} \;\middle|\; \overbrace{nL + H + \frac{t}{\alpha(K)} K}^{\xi} \text{ is nef} \right\}.$$

$r_L(n, H)$ is in $\mathbb{Z}_{>0}$. $r_L(n, H)$ non-dec wrt n

Indeed, If $n' > n$, then

$$n'L + H + \frac{r_L(n,H)}{\alpha(K)} K = \underbrace{\underbrace{(n'-n)L}_{\text{nef}} + \underbrace{nL + H + \frac{r_L(n,H)}{\alpha(K)} K}_{\text{nef}}}_{\text{nef.}}$$

Hence, we conclude that $r_L(n', H) \geq r_L(n, H)$.

On the other hand, we will see it is bounded above

Indeed, for any $\xi \in F_L \setminus \overline{NE}_{K\geq 0}$, we have

$$H\cdot\xi + \frac{r_L(n,H)}{\alpha(K)}\cdot K\cdot\xi \geq 0$$

$$r_L(n,H) \leq \alpha(K)\cdot\frac{H\cdot\xi}{-K\cdot\xi}.$$

So $r_L(n,H)$ is bounded above, is integral and non-dec.
This sequence stabilize for n large enough to $r_L(H)$

We define the divisor:

$$D(nL,H) := n\alpha(K)L + \alpha(K)H + r_L(H)K.$$

We claim that:

$$\underbrace{F_{D(nL,H)}}_{\text{orthogonal to } D(nL,H).} \subseteq \overline{NE}_{K<0} \cup \{0\}.$$

If $\xi\cdot D(nL,H) = 0$, then $(n\alpha(K)L + \alpha(K)H)\cdot\xi \geq 0$

so $\xi\cdot K < 0$, proving this claim.

We claim that for n large enough, we have.

$$(*) \quad F_{D(nL,H)} \subseteq F_L.$$

Let $\xi \in F_{D(nL,H)}$ with $\xi \notin F_L$. Then, we have that

$$\xi \cdot L \geq 0 \quad \text{and} \quad \xi \cdot (n\alpha(K)L + \alpha(K)H + r_L(H)K) = 0.$$

For $n' \gg n$, we have (this value will depend on $\alpha(K)$, H, $r_L(H)$ & K).

$$\xi \cdot (n'\alpha(K)L + \alpha(K)H + r_L(H)K) > 0.$$

Hence $\xi \notin F_{D(n'L,H)}$.

Since L is nef, we have $F_{D(n'L,H)} \subsetneq F_{D(nL,H)}$.

If $F_{D(n'L,H)} \subseteq F_L$, then we stop.

If not, we can iterate the above process to cut down

$\dim F_{D(n'L,H)}$ again.

This proves that $(*)$ eventually holds.

Hence, we have that:

$0 \neq F_{D(nL,H)} \subseteq F_L$ holds up to replacing n with a large multiple

Claim: for some H, $\dim F_{D(nL,H)} < \dim F_L$.

H_i basis F_L^*, the linear function

$$\left(nL + H_i + \frac{r_L(H_i)}{\alpha(K)} K\right)\Big|_{F_L}$$ they can't

be all zero, so $\dim F_{D(nL,H_i)} < \dim F_L$ for some i.

$F_{L'} \subseteq F_L$ $\quad F_{L'}$ is one dim.

$$\overline{NE} \quad \& \quad \overline{NE}_{K\geq 0} + \sum_{\dim F_L = 1} F_L$$

have the closure (verbatim from classic cone Thm).

Step 4: In this step, we prove that the F_L do not accumulate in $K_{<0}$.

Step 5: In this step, we prove that for $\varepsilon > 0$, we have the equality:

$$\overline{NE} = \overline{NE}_{K_X + \varepsilon H \geq 0} + \sum_{\substack{\text{finite} \\ F_L \cdot (K_X + \varepsilon H) < 0}}' F_L .$$

The cone Theorem follows from taking $\varepsilon \to 0$ in the above expression. (with some extra formal argument that we are omitting).

Step 6: We prove that if $F \subseteq \overline{NE}(X)$ is a $(K_X + \Delta)$-negative face, then there exists a nef Cartier divisor D so that $F_D = F$.

Let $\langle F \rangle$ be the linear span of F, $V \subseteq N_1(X)^*$ the set of linear functions vanishing identically on $\langle F \rangle$. Since the generators of F are spanned over $\mathbb{Q}$, then V is defined over $\mathbb{Q}$, take $\varepsilon > 0$ small enough so that

$$K_X + \Delta + \varepsilon H \text{ is negative on } F.$$

Since F is extremal, $\langle F\rangle \cap \overline{NE}(X) = F$. Thus

$$W_F := \overline{NE}(X)_{K_X+\Delta+\epsilon H \geq 0} + \sum_{\substack{\dim F_L = 1 \\ F_L \not\subseteq F}} F_L.$$

is a closed strictly convex cone int $\langle F\rangle$ at the origin.

Furthermore, $\overline{NE} = W_F + F$. Hence we can find a lattice point $g \in V$ so that $(g=0) \supseteq \langle F\rangle$ and

$(g=0) \cap W_F = 0$

Thus, we may find a Cartier divisor D which gives a supporting function of $F \subseteq NE(X)$.

Step 7: By assumption $-(K_X+\Delta)$ is positive on F.

$mD - (K_X+\Delta)$ is strictly positive on $\overline{NE}(X) \setminus \{0\}$

By bpf Theorem. $|mD|$ is bpf for $m \gg 0$

Let φ_F be the contraction associated by the Stein factorization to the bpf linear system $|mD|$.

Step 8: Since φ_F is not an isom (mD not ample), it must contract some curve C. Similar to the smooth case,

$$0 < -(K_X+\Delta).C \leq 2\dim(X).$$

Step 9: Let $X \xrightarrow{\varphi_F} Z$ be the contraction associated to F. We claim that any line bundle $\mathcal{L}$ on X such that $\mathcal{L}.F=0$ descends to Z i.e., there exists $\mathcal{L}_Z$ line bundle on Z so that $\mathcal{L} = \varphi_F^* \mathcal{L}_Z$.

Let D be a Cartier divisor supporting F. $W_F \subseteq \overline{NE}(X)$.

φ_F is defined by $|mD|$. So, both mD and $(m+1)D$ are pull-back of Cartier divisors on Z.

$$mD = \varphi_F^* D_1 \quad \text{Cartier.}$$

$$(m+1)D = \varphi_F^* D_2 \quad \text{Cartier.}$$

Thus, $D = (m+1)D - mD = \varphi_F^*(D_2 - D_1)$.

Hence D is the pull-back of a Cartier divisor on Z.

Now, let $\mathcal{L}$ with $\mathcal{L}.F=0$, then $\mathcal{L}+mD$ is also supporting F. Hence, $\mathcal{L}+mD = \varphi_F^* M_Z$ Cartier.

Set $\mathcal{L}_Z = \mathcal{O}_Z(M_Z - D_1)$.

$\square$